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Abbreviated key title: Asian j. appl. sci. eng.
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Generalization of Euler and Ramanujan's Partition Function

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ARTICLE INFO

Received: Sep 06, 2015
Accepted: Nov 06, 2015
Published: Nov 12, 2015

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crossref doi Prefix 10.18034

ABSTRACT

The theory of partitions has interested some of the best minds since the 18th century. In 1742, Leonhard Euler established the generating function of $P(n)$. Godfrey Harold Hardy said that Srinivasa Ramanujan was the first, and up to now the only, mathematician to discover any such properties of $P(n)$. In 1981, S. Barnard and J.M. Child stated that the different types of partitions of n in symbolic form. In this paper, different types of partitions of n are also explained with symbolic form. In 1952, E. Grosswald quoted that the linear Diophantine equation has distinct solutions; the set of solution is the number of partitions of n. This paper proves theorem 1 with the help of certain restrictions. In 1965, Godfrey Harold Hardy and E. M. Wright stated that the 'Convergence Theorem' converges inside the unit circle. Theorem 2 has been proved here with easier mathematical calculations. In 1853, British mathematician Norman Macleod Ferrers explained a partition graphically by an array of dots or nodes. In this paper, graphic representation of partitions, conjugate partitions and self-conjugate partitions are described with the help of examples.

Key Words: Ferrers and Young diagram, generating function, partitions, Ramanujan

Source of Support: None, **No Conflict of Interest:** Declared

How to Cite: Mohajan HK. 2015. **Generalization of Euler and Ramanujan's Partition Function** *Asian Journal of Applied Science and Engineering,* 4, 167-190.

INTRODUCTION

In this paper we have taken with the number of partitions of n with or without restrictions (Burn 1964). We have showed how to find the number of partition of n by using MacMahon's table (MacMahon 2005). In 1742, Euler also stated the series in the enumeration of partitions. In this paper we discuss how to generate the Euler's use of series in the enumeration of partitions. In 1952, Percy Alexander MacMahon also quoted the self-conjugate partitions of n. In this paper, MacMahon's self-conjugate partitions are explained with the help of array of dots or nodes.

A partition of n is a division of n into any number of positive integral parts. Then the sum of the integral parts or summands is n. The order of the parts and arrangement in a division of n

are irrelevant and the parts are arranged in descending order. Usually a partition of n is denoted by Greek letters π, λ or μ. We denote the number of partitions of n by $P(n)$. It is convenient to define $P(0) = 1$ and , $P(n) = 0$ for negative n.

At first glance, the material of partitions seems like mere a child's play. For example, consider the partitions of 4 as follows (Das and Mohajan 2015):

4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.

Hence, there are 5 partitions of the number 4, i.e., $P(4) = 5$. Gottfried Willhelm Leibnitz (1646–1716) was among the first mathematicians who contribute to the development of partitions (Griffin 1954). But the greatest contributions in the early stages of the partitions theory were due to great mathematician Leonhard Euler (1707–1783) in the mid-eighteenth century and yet it continues to reveal its mysteries (Andrews 1971).

We give some related definitions of $P(n)$, $P_m(n)$, $P^o(n)$, $P^d(n)$ and generating function (Das and Mohajan 2014a). We describe the different types of partitions of n in symbolic form and explain how to find the numbers of partitions of n into parts with or without restrictions. We explain elaborately the linear Diophantine equation $k_1a_1 + k_2a_2 + ... = n$, where k_1, k_2... are non-negative integers, has distinct solutions, the set of solutions is the number of partitions of n and prove theorem 1 with the help of terms $P_m(n)$, $P_{m-1}(n)$ and $P_m(n-m)$, and also prove theorem 2: The generating functions of the partitions converges inside the unit circle, i.e., $(1 + x + x^2 + ...)$ $(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)$ ... is convergent, if $|x| < 1$. We define graphic representation of partitions of numbers from 1 to 5 and give a brief survey of conjugated partitions and self-conjugate partitions with numerical examples respectively.

SOME RELATED DEFINITIONS AND DISCUSSION

Here we introduce some definitions related to the study following Andrews (1979), Barnard and Child (1967), Niven et al. (1991) and Das and Mohajan (2015).

Partition: In number theory a partition of a positive integer n, also called an integer partition is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. Let $A = \{a_1, a_2, ..., a_r, ...\}$ be a finite or infinite set of positive integers. If $a_1 + a_2 + ... + a_r = n$, with $a_r \in A$ ($r = 1$, 2, 3,…). Then we say that the sum $a_1 + a_2 + ... + a_r$ is a partition of n into parts belonging to the set A. So that, 3 + 2 + 1 is a partition of 6. If a partition contains p numbers, it is called a partition of n into p parts or shortly a p-partition of n. Hence, 9 = 4+2+1+1+1, and we can say that, 4+2+1+1+1 is a 5-partition of 9. The order of the parts is irrelevant, the parts to be arranged in descending order of magnitude.

Now explain how to find all the partitions of 7 as follows:

First take 7; then 6 allowed by 1; then 5 allowed by all the partitions of 2 (i.e., 2, 1+1); then 4 allowed by all the partitions of 3 (i.e., 3, 2+1, 1+1+1); then 3 allowed by all the partitions of 4, which contain no part greater than 3 (i.e., 3+1, 2+1+1, 1+1+1+1, 2+2); then 2 allowed by all the partitions of 5, which contain no part greater than 2 (i.e., 2+2+1, 2+1+1+1, 1+1+1+1+1); finally 1+1+1+1+1+1+1. Hence the complete set is;

7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+1+1, 3+1+1+1+1, 3+2+2, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1.

The partition 3+3 +3+3+1 might instead be written in the form (3, 3, 3, 3, 1) or in the even more compact form $(3^4,1)$ where the superscript denotes the number of repetitions of a term. Hence, 2+2+2+1, 2+1+1+1+1+1 and 1+1+1+1+1+1+1 can be written as, 2^3+1, $2+1^5$ and 1^7 respectively.

The numbers of partitions of n are as table 1.

Table 1: The number of the partitions $P(n)$ for n = 0,1, 2, 3, 4, 5.

n	Type of partitions	$P(n)$
0	0	1
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	5
5	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1	7

In the light of table 1 we can write an expression for $P(n)$ as;

$$P(0)+P(1)x+P(2)x^2+P(3)x^3+P(4)x^4+P(5)x^5+\ldots$$

$$=1+1.x+2.x^2+3.x^3+5.x^4+7.x^5+\ldots$$

$$=\left(1+x+x^2+\ldots\right)\left(1+x^2+x^4+\ldots\right)\left(1+x^3+x^6+\ldots\right)\ldots$$

$$=\frac{1}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots}$$

$$=\prod_{i=1}^{\infty}\frac{1}{1-x^i}$$

$$=\sum_{n=0}^{\infty}P(n)x^n.$$

In 1952, a major in the British Royal Artillery, Percy Alexander MacMahon used Euler's result to establish a table of $P(n)$ for the first 200 values of n, which he found to be;

$$P(200) = 3972999029388.$$

He did not count the partitions one-by-one as follows:

$$200 = 199+1 = 198+2 = 198+1+1 = 197+3 = \ldots$$

Instead, he used classical formal power series identities due to Euler. After a distinguished career with the Royal Artillery in Madras (India) and as an instructor at the Royal Military

Academy, Woolwich, MacMahon at the age of 58 went up to Cambridge University to pursue research in combinatorial number theory. He was elected a member of St. John's College and served as president of the London Mathematical Society and of the Royal Astronomical Society (Tattersall 1999).

$\boldsymbol{P(n)}$**:** The number of partitions of n, also called the partition function. In number theory, the partition function $P(n)$ represents the number of possible partitions of a natural number n, which is to say the number of distinct ways of representing n as a sum of natural numbers (with order irrelevant). By convention we have $P(0) = 1$, $P(n) = 0$ for n negative. Indian great mathematician Srinivasa Ramanujan was perhaps the first mathematician who seriously investigate the properties of partition function $P(n)$. He established a formula for $P(n)$, one which describes the exceptional rate of growth suggested by the table 2 (Ono 2009).

Table 2: The value of the partition function $P(n)$ for $n = 0,1,\ldots,10000$.

n	$P(n)$
0	1
1	1
2	2
3	3
4	5
5	7
.	.
.	.
.	.
50	204226
.	.
.	.
.	.
100	190569292
.	.
.	.
.	.
200	3972999029388
.	.
.	.
.	.
1000	24061467864032622473692149727991 or 2.40615×10^{31}
.	.
.	.
.	.
10000	361672513256 ... 906916435144 or 3.61673×10^{106}

As of June 2013, the largest known prime number that counts a number of partitions is $P(120052058)$, with 12,198 decimal digits.

Together with Hardy, Ramanujan gave a remarkable asymptotic formula in 1917 (Hardy and Ramanujan 1917a, b) as;

$$P(n) \approx \frac{1}{4\sqrt{3}\,n} \exp\left(\pi\sqrt{2n/3}\right) \text{ as } n \to \infty .$$

Hardy and Ramanujan obtained an asymptotic expansion with this approximation as the first term;

$$P(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{v} \sqrt{k} A_k(n) \frac{d}{dn} \exp\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right), \tag{1}$$

where $$A_k(n) = \sum_{0 \le m \le k;(m,k)=1} \exp\left[\pi i\left(s(m,k) - \frac{2mn}{k}\right)\right].$$

Here, $(m, n) = 1$ implies that the sum should occur only over the values of m that are relatively prime to n. The function $s(m, k)$ is a Dedekind sum. This Hardy and Ramanujan result was perfected by Hans Adolph Rademacher, a number theorist at the University of Pennsylvania, two decades later to obtain a formula which is so accurate that it can be used to compute individual values of $P(n)$ (Rademacher 1943, 1973). Rademacher found an expression that, when rounded to the nearest integer, equal to $P(n)$. In 1937, he was able to improve on Hardy and Ramanujan's results by providing a convergent series expression for $P(n)$ as;

$$P(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)\right), \tag{2}$$

where $$A_k(n) = \sum_{0 \le m \le k;(m,k)=1} \exp\left[\pi i\left(s(m,k) - \frac{2mn}{k}\right)\right].$$

He defined explicit functions $T_k(n)$ such that for all positive n;

$$P(n) = \sum_{k=1}^{\infty} T_k(n). \tag{3}$$

Hardy called it "one of the rare formulae which are both asymptotic and exact." The function $T_1(n)$ alone gives the Hardy-Ramanujan asymptotic formula;

$$P(n) \sim \frac{1}{4n\sqrt{3}} . e^{\pi\sqrt{2n/3}} . \tag{4}$$

Moreover, Rademacher computed precisely the error incurred by truncating this series after Q terms. In particular, there exist explicit constants A and B such that (Ahlgren and Ono 2001);

$$\left| P(n) - \sum_{k=1}^{A\sqrt{n}} T_k(n) \right| < \frac{B}{n^{1/4}}. \tag{5}$$

For n = 200, he found the approximation value;

$$P(200) \approx 3972999029388.004,$$

which is nicely compares with the exact value in the table 2.

Techniques for implementing the Hardy-Ramanujan-Rademacher formula efficiently on a computer are discussed by Fredrik Johansson, where it is shown that $P(n)$ can be computed in softly optimal time $O(n^{0.5+\varepsilon})$, which is softly optimal since $P(n)$ has $\Theta(n^{0.5})$ bits (Johansson 2012). The largest value of the partition function computed exactly is $P(10^{20})$, which has slightly more than 11 billion digits. In theory, $P(10^{20})$ is therefore $10^{0.5} \approx 3.16$ times more expensive to compute than $P(10^{19})$. In fact, the computation of $P(10^{20})$ only required 110 hours and 130 GB of memory, comparable to the 100 hours and 150 GB used for the computation of $P(10^{19})$ done in 2011, due to improvement of computer technology. $P(10^{20})$ starting with

18381765083448826436460575151963949703661288601871338187949218306809161793

5585192260508725895357972 1...

and ending with

... 959766125017460247986152430226200195597077070328758246298447232570089919

8905833521126231756788091448.

To compute $P(10^{21})$, which overflows around 40 billion digits, need to be developed the technology of the computer. Computation of $P(10^{24})$ will be about 1.1 trillion digits! (Johansson 2014).

$P^d(n)$: The number of partitions of n into distinct parts. But $P^d(n)$ is sometimes denoted by $Q(n,*,*)$ where asterisk '*' means that no restriction is placed on the number or the nature of the parts (Das and Mohajan 2014b).

The number of partitions of n into distinct parts is denoted by $P^d(n)$. Some of such partitions are given in table 3.

Table 3: The value of $P^d(n)$ for n = 1, 2, 3, 4.

n	Type of partitions	$P^d(n)$
1	1	1
2	2	1
3	3, 2+1	2
4	4, 3+1	2

We can write an expression for $P^d(n)$ as;

$$1+P^d(1)x+P^d(2)x^2+P^d(3)x^3+P^d(4)x^4+\ldots$$

$$=1+1.x+1.x^2+2.x^3+2.x^4+\ldots$$

$$=(1+x)(1+x^2)(1+x^3)\ldots$$

$$=\prod_{n=1}^{\infty}(1+x^n)$$

$$=1+\sum_{n=1}^{\infty}P^d(n)x^n.$$

$P^o(n)$: The number of partitions of n into odd parts. Some of such partitions are given in table 4.

Table 4: The value of $P^o(n)$ for n = 1, 2, 3, 4.

n	Type of partitions	$P^o(n)$
1	1	1
2	1+1	1
3	3, 1+1+1	2
4	3+1, 1+1+1+1	2

We can write an expression for $P^o(n)$ as;

$$1+P^o(1)x+P^o(2)x^2+P^o(3)x^3+P^o(4)x^4+\ldots$$

$$=1+1.x+1.x^2+2.x^3+2.x^4+\ldots$$

$$=\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}$$

$$=1+\sum_{n=1}^{\infty}P^o(n)x^n.$$

$P_m(n)$: The number of partitions of n into parts no larger than m.

The number of partitions of n having only the numbers 1 and/or 2 as parts is denoted by $P_2(n)$. Some of such partitions are given in table 5.

Table 5: The value of $P_2(n)$ for n = 1, 2, 3, 4.

n	Type of partitions	$P_2(n)$
1	1	1
2	2, 1+1	2
3	2+1, 1+1+1	2
4	2+2, 2+1+1, 1+1+1+1	3

We can write an expression for $P_2(n)$ as;

$$1+P_2(1)x+P_2(2)x^2+P_2(3)x^3+P_2(4)x^4+...$$

$$=1+1.x+2.x^2+2.x^3+3.x^4+\dots$$

$$=(1+x+x^2+...)(1+x^2+x^4+...)...$$

$$=\frac{1}{(1-x)(1-x^2)\ ...}$$

$$=1+\sum_{n=1}^{\infty}P_2(n)x^n\ .$$

We can define $P_2(n)$ in another way as follows:

The number of partitions of n into at most two parts is $P_2(n)$ and can be expressed as table 6.

Table 6: The value of $P_2(n)$ for n = 1, 2, 3, 4.

n	Type of partitions	$P_2(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1	2
4	4, 3+1, 2+2	3

We can write an expression for $P_2(n)$ as;

$$1+P_2(1)x+P_2(2)x^2+P_2(3)x^3+P_2(4)x^4+...$$

$$=1+1.x+2.x^2+2.x^3+3.x^4+\dots$$

$$=(1+x+x^2+...)(1+x^2+x^4+...)...$$

$$=\frac{1}{(1-x)(1-x^2)\ ...}$$

$$=1+\sum_{n=1}^{\infty}P_2(n)x^n\ .$$

That is, we can say $P_2(n)$ is the number of partitions of n into parts not exceeding 2 or the number of partitions of n having only the numbers 1 and/or 2 as parts.

The number of partitions of n having only the numbers 1, 2 and/or 3 as parts is denoted by $P_3(n)$. Some of such partitions are given in table 7.

Table 7: The value of $P_3(n)$ for $n = 1, 2, 3, 4$.

n	Type of partitions	$P_3(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	3+1, 2+2, 2+1+1, 1+1+1+1	4

We can write an expression for $P_3(n)$ as;

$$1+P_3(1)x+P_3(2)x^2+P_3(3)x^3+P_3(4)x^4+...$$

$$=1+1.x+2.x^2+3.x^3+4.x^4+\dots$$

$$=\frac{1}{(1-x)(1-x^2)(1-x^3)...}$$

$$=1+\sum_{n=1}^{\infty}P_3(n)x^n.$$

We can define $P_3(n)$ in another way as follows:

The number of partitions of n into at most two parts is $P_3(n)$ and can be expressed as table 8.

Table 8: The value of $P_3(n)$ for $n = 1, 2, 3, 4$.

n	Type of partitions	$P_3(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 2+2, 2+1+1	4

We can write an expression for $P_3(n)$ as;

$$1+P_3(1)x+P_3(2)x^2+P_3(3)x^3+P_3(4)x^4+...$$

$$=1+1.x+2.x^2+3.x^3+4.x^4+\dots$$

$$=(1+x+x^2+...)(1+x^2+x^4+...)(1+x^3+x^6+...)...$$

$$=\frac{1}{(1-x)(1-x^2)(1-x^3)...}$$

$$=1+\sum_{n=1}^{\infty}P_3(n)x^n.$$

Therefore, we can say $P_3(n)$ is the number of partitions of n into parts not exceeding 3 or the number of partitions of n having only the numbers 1, 2 and/or 3 as parts. Generally we can say that $P_m(n)$ is the number of partitions of n into parts not exceeding m. Hence,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^m)}=1+\sum_{n=1}^{\infty}P_m(n)x^n .$$

GENERATING FUNCTION

In mathematics, a generating function is a formal power series in one intermediate, whose coefficients encode information about a sequence of numbers b_n that is indexed by the natural numbers. Generating functions were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem (Doubilet et al. 1972). Wilf (1994) defined the generating function as "*A generating function is a clothesline on which we hang up a sequence of numbers for display*."

The power series;

$$f(x)=1+\sum_{n=1}^{\infty}b_n x^n , \tag{6}$$

is called the generating function of the sequence $\langle b_n \rangle$. The generating function for $P(n)$ was found by Leonhard Euler in 1742 and can be written as follows:

$$\sum_{n=0}^{\infty}P(n)x^n=\prod_{i=1}^{\infty}\frac{1}{1-x^i} . \tag{7}$$

A partition of n may be represented graphically by an array of dots or nodes. The parts of a partition of n may be even positive integral parts, odd positive integral parts, distinct parts etc. i.e., the parts have various restrictions. A graceful tool in the study of partitions is the Ferrers diagram. The *conjugate* of a partition is obtained by interchanging the rows and the columns of the Ferrers diagram (discuss later).

DIFFERENT TYPES OF PARTITIONS OF *N*

The number of partitions in any type is denoted by a symbol of the form $P(n, ,)$, where the number and the nature of the parts respectively indicated in the first and second places following n (Burn 1964, Hardy and Wright 1965). In the first place, p means that there are p parts and $\leq, p$ means that the number of parts does not exceed p. In the second place, q means that the greatest number of parts is q and $\leq, q$ means that no part exceeds q. An asterisk '$*$' means that no restriction is placed on the number or the nature of the parts. If the parts are to be unequal, then Q is used instead of P.

Thus, $P(n, p, q)$ is the number of partitions of n into p parts, the greatest of which is q. $P(n, \leq p, *)$ is the number of partitions of n into p or any smaller number of parts.
$Q(n, *, \leq q)$ is the number of partitions of n into unequal parts, none of which exceeds q.
Here $P(n, p, *)$ is the number of p-partitions of n.

Example 1: We shall find the number of partitions of 11 into 4 parts. The partitions of 11 into 4 parts are 8+1+1+1, 7+2+1+1, 6+3+1+1, 6+2+2+1, 5+4+1+1, 5+2+2+2, 5+3+2+1, 4+4+2+1, 4+3+2+2, 4+3+3+1, 3+3+3+2; and their number is 11.

$\therefore P(11, 4, *) = 11.$ (8)

NUMBER OF PARTITIONS OF *N* INTO PARTS WITH OR WITHOUT RESTRICTIONS

In this section we explain the number of partitions of n into parts with or without restrictions. Let $f(x)$ be a generating function, whose denominator is a product of infinite factors $(1-x)$, $(1-x^2)$, $(1-x^3)$, ... ∞, and it can be written as follows;

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ ...\ \infty}$$

$$= (1+x+x^2+...)(1+x^2+x^4+...)\ ...$$

$$= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+...$$

But we have; $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 7$, $P(6) = 11$, …

So the expansion can be written as:

$$f(x) = \frac{1}{(1-x)(1-x^2)...}$$

$$= 1+P(1)x+P(2)x^2+P(3)x^3+P(4)x^4+P(5)x^5+...$$

$=1+\sum_{n=1}^{\infty} P(n)\, x^n$, it is convenient to define $P(0) = 1$.

Therefore, $P(n)$ is the coefficient of x^n in the expansion of $\frac{1}{(1-x)(1-x^2)...}$ and is the number of partitions of n into parts without restrictions.

We have already observed each partition is completely determined by the set of non-negative integers $k_1, k_2, ..., k_r, ...$; hence the number $P(n)$ of partitions of n is precisely the number of distinct solutions of the linear Diophantine equation (Grosswald 1952);

$$k_1a_1 + k_2a_2 + ... + k_ja_j + ... = n, \tag{9}$$

where $a_i \in A$ (the set of positive integers). We illustrate it with an example linear Diophantine equation (9); hence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, … If $n = 5$ we can write from (9) as;

$$k_1a_1 + k_2a_2 + k_3a_3 + k_4a_4 + k_5a_5 = 5$$
$$k_1.1 + k_2.2 + k_3.3 + k_4.4 + k_5.5 = 5$$

Now, $0.1+0.2+0.3+0.4+1.5=5$ has solution $(0,0,0,0,1)$,
$1.1+0.2+0.3+1.4+0.5=5$ has solution $(1,0,0,1,0)$,
$0.1+1.2+1.3+0.4+0.5=5$ has solution $(0,1,1,0,0)$,
$2.1+0.2+1.3+0.4+0.5=5$ has solution $(2,0,1,0,0)$,
$1.1+2.2+0.3+0.4+0.5=5$ has solution $(1,2,0,0,0)$,
$3.1+1.2+0.3+0.4+0.5=5$ has solution $(3,1,0,0,0)$, and
$5.1+0.2+0.3+0.4+0.5=5$ has solution $(5,0,0,0,0)$

and their corresponding partitions are; 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1 respectively. Hence we have $P(5)=7$ solutions. In symbol; $P(5)=\sum 1$; where $\sum k_i a_i = 5$, and $i=\overline{1-5}$, $a_i \in A$, therefore $A=\{a_1, a_2, \ldots\}$ set of positive integers, $k_1, k_2, \ldots$ are non-negative integers. Generally we can conclude that, $P(n)=\sum 1$; where $\sum k_i a_i = n$, and $i=\overline{1-n}$, $a_i \in A$, therefore $A=\{a_1, a_2, \ldots\}$ set of positive integers, $k_1, k_2, \ldots$ are non-negative integers.

Generating Functions

Let $f_m(x)$ be a generating function, whose denominator is the product of the factors $(1-x), (1-x^2), \ldots, (1-x^m)$, and it can be written as in the form;

$$f_m(x)=\frac{1}{(1-x)(1-x^2)\ldots(1-x^m)}$$
$$=(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ \ldots\ (1+x^m+x^{2m}+\ldots)$$
$$=1+P_1(1)x+P_2(2)x^2+\ldots$$
$$=1+\sum_{n=1}^{\infty} P_m(n)\,x^n,$$

where $P_m(n)$ is the number of partitions of n into parts not exceeding m. From above expansions we have the following remarks and a theorem.

Remark 1: $P_m(n)=P(n)$, if $n \le m$.

This is obtained from the definition of partition function $P(n)$ and $P_m(n)$.

Example 2: If $n=5$ and $m=7$.

There are 7 partitions of 5 into parts with no restriction and these are given below:

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

Thus the number of such partitions is, $P(5)=7$. But we can say that the number of partitions of 5 into parts not exceeding 7 is equal to the number of partitions of 5 into parts not exceeding 5 and these partitions are shown above. Hence, $P_7(5)=P_5(5)=7$.

Thus, $P(5) = P_7(5) = P_5(5)$. Generally we can write, $P_m(n) = P(n)$, if $n \le m$.

Remark 2: $P_m(n) \le P(n)$, for all $n > 0$.

This is also obtained from the definition of partitions functions $P(n)$ and $P_m(n)$.

Example 3: If n = 6 and m = 4, there are 11 partitions of 6 into parts with no restriction and these are given below:

6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.

Thus the number of such partitions is $P(6)$ = 11. Again the number of partitions of 6 into parts none of which exceeds 4 is obtained as follows:

4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.

So the number of such partitions is $P_4(6) = 9$. From above two cases, we get, $P_4(6) < P(6)$.

Generally we can write; $P_m(n) < P(n)$, if $m < n$.

If $m \le n$, $P_m(n) \le P(n)$, for all $n > 0$ and $n \ge m > 1$.

Theorem 1: $P_m(n) = P_{m-1}(n) + P_m(n-m)$, if $n \ge m > 1$.

Proof: We know that the partition of n counted by $P_m(n)$ either has or does not have a part equal to m. The partitions of the second sort are counted by $P_{m-1}(n)$. The partitions of the first sort are obtained by adding the partitions, where as each partition of n–m into parts less than or equal to m, and there are $P_m(n-m)$ in numbers. That is the sum of $P_{m-1}(n)$ and $P_m(n-m)$ is equal to $P_m(n)$. If n = m the term $P_m(n-m) = 1$ counts the single partition. Therefore,

$$P_m(n) = P_{m-1}(n) + P_m(n-m), \text{ if } n \ge m > 1. \qquad \blacksquare$$

Example 4: If n = 8 and m = 3, the partitions of 8 into parts which do not exceed 3 are given below:

3+3+2, 3+3+1+1, 3+2+1+1+1, 3+1+1+1+1+1, 3+2+2+1, 2+2+2+2, 2+2+2+1+1, 2+2+1+1+1+1, 2+1+1+1+1+1+1, 1+1+1+1+1+1+1+1.

Thus the number of such partitions is $P_3(8) = 10$. The partitions of 8 into parts, none of which exceeds 2 are given below:

2+2+2+2, 2+2+2+1+1, 2+2+1+1+1+1, 2+1+1+1+1+1+1, 1+1+1+1+1+1+1+1.

So, the number of such partitions is $P_2(8) = 5$. Finally, the partitions of 5 into parts not exceeding 3 are given as follows:

3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

So the number of such partitions is $P_3(5)=5$.

From above three cases, we get;

$P_3(8)=10 = 5+5 = P_2(8)+P_3(5)$.

From above theorem, we can consider the following comparisons:

i) $P_m(n-m)=1$, when $m = n$, there is just one partition of n with greatest part is m.

ii) $P_m(n-m)=0$, then $n < m$, then $P_m(n)=P_{m-1}(n)$.

Now we introduce a theorem about convergence as follows:

Theorem 2: The generating functions of the partition functions converge inside the unit circle.

Proof: We know that; $\frac{1}{(1-x)(1-x^2)(1-x^3)\ \dots}$ is a generating function of the partition function $P(n)$ and it can be written as; $\frac{1}{(1-x)(1-x^2)(1-x^3)\ \dots}=\sum_{n=0}^{\infty}P(n)x^n$. The number of partitions of n with or without restrictions is a non-negative integer also the number of partitions can only decrease, if restrictions are added. The series;

$1+x+x^2+\dots+x^{n-1}=\frac{1-x^n}{1-x}$, if $|x|<1$ and $n\to\infty$, So $x^n\to 0$, then the above series becomes;

$1+x+x^2+\dots=\frac{1}{1-x}$, which is convergent, if $|x|<1$.

Hence the series $1+x+x^2+\dots$ is convergent, if $|x|<1$ and when convergent, its sum is $\frac{1}{1-x}$. Similarly we can easily verify that the series $(1+x+x^2+\dots)(1+x^2+x^4+\dots)$ is also convergent, if $|x|<1$ and its sum is $\frac{1}{(1-x)(1-x^2)}$. Thus we can say that the series;

$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\ \dots$ is convergent, if $|x|<1$ and when convergent its sum is equal to $\frac{1}{(1-x)(1-x^2)(1-x^3)\ \dots}$. Hence, we can express that the generating functions of the partition functions converge inside the unit circle. ■

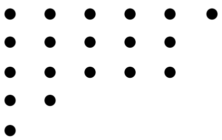
DIAGRAMMATIC REPRESENTATION OF PARTITIONS

There are two common diagrammatic methods to represent partitions: i) Ferrers diagrams, named after Norman Macleod Ferrers, and ii) Young diagrams, named after the British mathematician Alfred Young.

FERRERS DIAGRAM

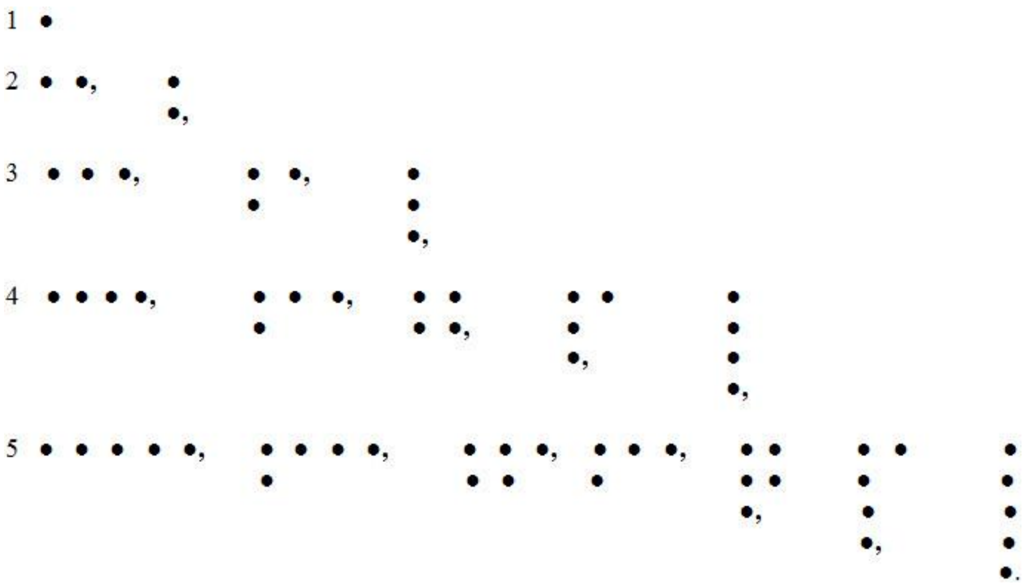
A partition of n can be represented graphically. In 1853, Norman Macleod Ferrers communicated to James Joseph Sylvester an ingenious method for representing partitions. Graphs of partitions were conceived by Ferrers and first appeared in print in a paper by J. J. Sylvester in 1853 (Sylvester 1882). A partition of n may be represented graphically by an array of dots or nodes. A graceful tool in the study of partitions is the Ferrers diagram. The conjugate of a partition is obtained by interchanging the rows and the columns of the Ferrers diagram.

If $n = a_1 + a_2 + ... + a_r$, we may presume that $a_1 \geq a_2 \geq ... \geq a_r$. Then the graph of the partition is the array of points having a_1 points in the top row, a_2 in the next row, and so on down to a_r in the bottom row,



Hence, 6+5+5+2+1 = 19.

Now we represent a list of Ferrers graph as follows:



Conjugate Partitions

Two partitions are said to be conjugate, when the graph of one is obtained by interchanging the rows and columns of the other (Andrews 1967). The columns of the graph

• • • •
• • •
•
•

give the partition 4+3+1+1.

The rows of the same graph give the partition, 4+2+2+1. Interchanging rows and columns converts;

• • • • to • • • •
• •   • • •
• •   •
•   •

such pairs of partitions are said to be conjugate.

Self-Conjugate Partitions

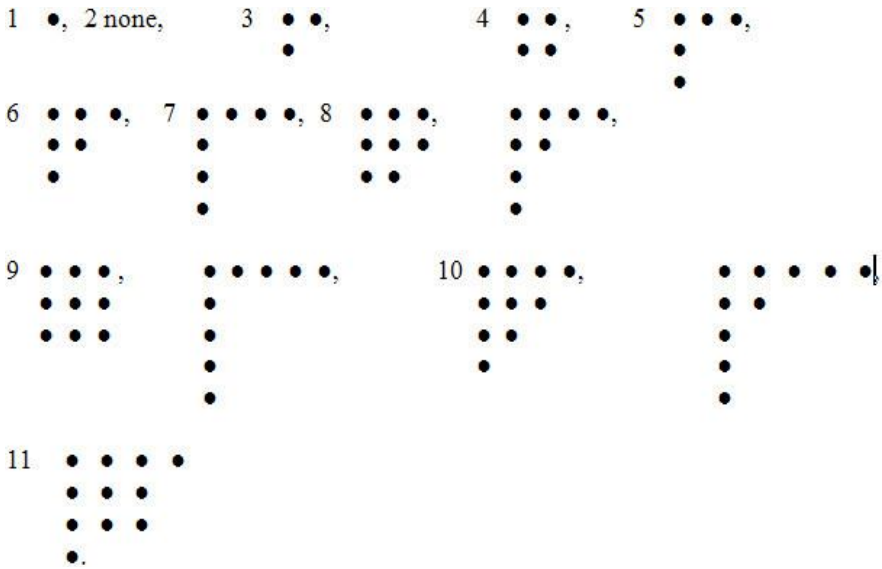
If a partition has no interchange the rows and corresponding columns, such partition is called the self-conjugate partition.

Thus • • • is a self-conjugate partition
• • •
• • •

i.e., • • • is a partition of 9, which has no conjugate to any other partition of 9.
• • •
• • •

Now we will set some examples of conjugate partitions or self-conjugate partitions.

Example 5: There is a list of all the partitions of numbers up to 11, which are not conjugate to any other partition.



Example 6: Every odd number has at least one self-conjugate partition. $2n + 1$ may be partitioned into

$$(n+1) + \underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}$$

for which the graph is

n nodes

● ● ● . . . ●
●
.
.
.
n nodes ●
●.

Example 7: If a number has a self-conjugate partition, it must have at least one partition into distinct odd parts.

Thus

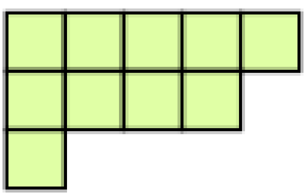
● ● ● ● ● ● 11
● ● ● ● + 5
● ● ● + 1
● ●
●
●.

Example 8: If a number has a partition into distinct odd parts, it must have a self-conjugate partition. The odd numbers each give L-shaped arrange as in Example 6 and if they are distinct and are fitted together in order of size as in Example 7, the resulting array is the graph of a partition. Since each part added is self-conjugate and is added to a self-conjugate graph, the result is a self-conjugate graph.

YOUNG DIAGRAM

Young diagrams, named after the British mathematician Alfred Young, turn out to be extremely useful in the study of symmetric functions and group representation theory; in particular, filling the boxes of Young diagrams with numbers obeying various rules leads to a family of objects called Young tableaux, and these tableaux have combinatorial and representation-theoretic significance (Andrews 1976).

Rather than representing a partition with dots, as in the Ferrers diagram, the Young diagram uses boxes or squares. Thus, the Young diagram for the partition 5 + 4 + 1 is,



while the Ferrers diagram for the same partition is;

● ● ● ● ●
● ● ● ●
●

as a type of shape made by above squares joined together, Young diagrams are a special kind of polymino (Josuat-Vergès 2010).

RESTRICTED PARTITION FUNCTIONS

We also discuss about the restricted partition functions which are also used in Euler's series in the enumeration of partitions. The generating function for $P(n, p, \leq q)$ is of the form;

$$\frac{1}{(1-zx)(1-zx^2)\ldots(1-zx^q)} = 1+\sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, \leq q) \right\}. \qquad (10)$$

It is convenient to define $P(n, p, \leq q) = 0$, if $n < p$. The coefficient $P(n, p, \leq q)$ is the number of partitions of n into p parts, none of which exceeds q.

Example 9: If $q = 3$, the above function becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)}$$

$$= 1 + zx + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(2z^2+z^3+z^4) + \ldots$$

$$= 1+\sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, \leq 3) \right\}.$$

From (10) we have;

$$= 1+\sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, p, \leq q) \right\}$$

$$= \frac{1}{(1-zx)(1-zx^2)\ldots(1-zx^q)}$$

$$= 1+\sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, \leq q, p) \right\}.$$

Equating the coefficient of $x^n z^p$ from both sides, we get the following remarks:

Remark 3: $P(n, p, \leq q) = P(n, \leq q, p)$.

If $q \to \infty$ in (10), such as $\lim_{q\to\infty} x^q = 0$, $|x| < 1$ then (10) becomes;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots}$$
$$=1+zx+x^2(z+z^2)+x^3(z+z^2+z^3)+x^4(2z^2+z^3+z^4)+\ldots$$
$$=1+\sum_{n=1}^{\infty}x^n\left\{\sum_{p=1}^{n}z^pP(n,p,*)\right\}. \qquad (11)$$

where the coefficient $P(n,p,*)$ is the number of partitions of n into p parts. From (11) we have;

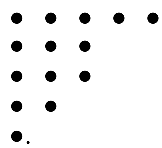
$$=1+\sum_{n=1}^{\infty}x^n\left\{\sum_{p=1}^{n}z^pP(n,p,*)\right\}$$
$$=1+zx+x^2(z+z^2)+x^3(z+z^2+z^3)+x^4(2z^2+z^3+z^4)+\ldots$$
$$=1+\sum_{n=1}^{\infty}x^n\left\{\sum_{p=1}^{n}z^pP(n,*,p)\right\}.$$

Equating the coefficient of x^nz^p from both sides, we get the following remark:

Remark 4: $P(n,p,*)=P(n,*,p)$.

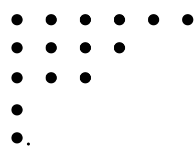
Proof: We establish a one-to-one correspondence between the partitions enumerated by $P(n,p,*)$ and those enumerated by $P(n,*,p)$. Let $n=a_1+a_2+\ldots+a_p$ be a partition of n into p-parts. We transfer this into a partition of n having largest part p and can represent a partition of 15 graphically by an array of dots or nodes such as,

A



The dots in a column correspond to a part. Thus *A* represents the partition 6+4+3+1+1 of 15. We can also represent *A* by transposing rows and columns in which case it would represent the partition graphically as conjugate of *A*.

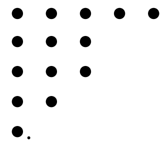
B = Conjugate of *A*



The dots in a column correspond to a part, so that it represents the partition 5+3+3+2+1+1 of 15. Such pair of partitions are said to be conjugate. The number of parts at the 1st one portion is equal to the largest part of the 2nd one partition, so that our corresponding is one-to-one.

Conversely, we can represent the partition B = conjugate of A, by transposing rows and columns, in which case it would represent the same partition like A, so we can say that the largest part of the partition is equal to the number of parts of the partition, then our corresponding is onto, i.e., the number of partitions of n into p-parts is equal to the number of partitions of n having largest part p. Consequently,

A



$$P(n, p,*) = P(n,*, p). \blacksquare$$

Example 10: The list of partitions of 8 into 4 parts is given below:

5+1+1+1 = 4+2+1+1 = 3+3+1+1 = 3+2+2+1 = 2+2+2+2.
The number of such partitions is 5, i.e., $P(8, 4, *) = 5$.

Again the list of partitions of 8 having largest part 4 is given below:

4+4 = 4+3+1 = 4+2+1+1 = 4+1+1+1+1 = 4+2+2.

So the number of such partitions is 5, i.e., $P(8, *, 4) = 5$.

Here, 4+4, 4+3+1, 4+2+1+1, 4+1+1+1+1, and 4+2+2 are the conjugate partitions of 2+2+2+2, 3+2+2+1, 4+2+1+1, 5+1+1+1 and 3+3+1+1 respectively. Thus the number of partitions of 8 into 4 parts is equal to the number of partitions of 8 into parts, the largest of which is 4 i.e., $P(8,4,*) = P(8,*,4)$.

Generally, we can say that the number of partitions of n into p parts is the same as the number of partitions of n having largest part p. The generating function for $P(n, \le p, \le q)$ is of the form;

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^q)}$$

$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^{n} z^p P(n, \le p, \le q) \right\} \qquad (12)$$

where the coefficient $P(n, \le p, \le q)$ is the number of partitions of n into p or any smaller number of parts, none of which exceeds q.

Example 11: If $q = 2$ then the function (12) becomes;

$$\frac{1}{(1-z)(1-zx)(1-zx^2)}$$

$$=1+\left(z+z^2+z^3+...\right)+x\left(z+z^2+z^3+...\right)+x^2\left(z+2z^2+...\right)+x^3\left(z^2+2z^3+...\right)+...$$

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n,\leq p,\leq 2\right)\right\}.$$

If $q\rightarrow\infty$, such as $\lim_{q\rightarrow\infty}x^q=0$, $|x|<1$ then (12) becomes;

$$\frac{1}{(1-z)(1-zx)(1-zx^2)...}$$

$$=1+\left(z+z^2+z^3+...\right)+x\left(z+z^2+z^3+...\right)+x^2\left(z+2z^2+...\right)+x^3\left(z^2+2z^3+...\right)+...$$

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n,\leq p,*\right)\right\} \qquad (13)$$

where the coefficient $P(n,\leq p,*)$ is the number of partitions of n into p or any smaller number of parts. From (13) we have;

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n,\leq p,*\right)\right\}$$

$$=1+\left(z+z^2+z^3+...\right)+x\left(z+z^2+z^3+...\right)+x^2\left(z+2z^2+...\right)+x^3\left(z^2+2z^3+...\right)+...$$

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n+p,p,*\right)\right\}.$$

Equating the coefficient of x^nz^p from both sides, we get the following remark.

Remark 5: $P(n,\leq p,*)=P(n+p,p,*)$.

Again from (13) we have;

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n,\leq p,*\right)\right\}$$

$$=1+\left(z+z^2+z^3+...\right)+x\left(z+z^2+z^3+...\right)+x^2\left(z+2z^2+...\right)+x^3\left(z^2+2z^3+...\right)+...$$

$$=1+\sum_{n=0}^{\infty}x^n\left\{\sum_{p=1}^{\infty}z^pP\left(n,*,\leq p\right)\right\}.$$

Equating the coefficient of x^nz^p from both sides, we get the following remark:

Remark 6: $P(n, \leq p, *) = P(n, *, \leq p)$.

Proof: The generating function for $P_p(n)$ is;

$$f_p(x) = \frac{1}{(1-x)(1-x^2)\ldots(1-x^p)}$$

$$= 1 + \sum_{n=1}^{\infty} P_p(n)x^n .$$

Suppose that $0 < x < 1$, so that the product is convergent. Now we consider the product;

$$(1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\ldots(1 + x^p + x^{2p} + \ldots) .$$

Any term out of the 1st, 2nd, …, pth factors may be represented by; x^{α}, $x^{2\beta}$, $x^{3\gamma}$,…, $x^{p\theta}$, where $\alpha, \beta, \gamma, \ldots, \theta$ are any of the numbers 0, 1, 2, …, p.

Thus, $\alpha = 0$, $\beta = 1$, $\gamma = 2$, the partition,

$$2 + \underbrace{3 + 3}_{2 \text{ times}}$$

=2 + 2.3 is one partition of 8 and α =1, β=2, γ= 1,

the partition;

$$1 + \underbrace{2 + 2}_{2 \text{ times}} + 3$$

= 1+ 2.2 + 3 is also one of 8.

If the product of these terms is x^n, we have $\alpha + 2\beta + 3\gamma + \ldots + p\theta = n$, which is not unique. Hence the coefficient of x^n in the product is the number of partitions in which n can be obtained by adding any representation of the numbers 1,2,…,p repetitions being allowed. Now the coefficient of x^n in the product is the number of partitions of n into at most p parts.

But in remark 5 we have discussed that the number of partitions of n into p parts is equal to the number of partitions of n into parts, the largest of which is p. So we can say that the number of partitions of n into at most p parts is equal to the number of partitions of n into parts which do not exceed p. Hence,

$$P(n, \leq p, *) = P(n, *, \leq p). \qquad \blacksquare$$

CONCLUSION

In this article, we have established the number $P(n)$ of partitions of n is the number of distinct solutions of the linear Diophantine equation, where n is any positive integer. We have verified theorem 1 when n = 8 and m = 3, and have proved the theorem 2 with the help of convergence condition $|x| < 1$. We have found a partition which has no interchange the rows and corresponding columns; such a partition is called the self-conjugate partition, and has established every odd number has at least one self-conjugate partition. We have provided some remarks and also set some examples to make the paper interesting to the readers.

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